Probability

Definitions:

Experiment - any happening for which the result is uncertain

Outcome – the possible result of the experiment

Sample space – the set of all possible outcomes of the experiment

Event - any subset of the sample space

Example: Find the sample space of the experiment of one fair coin being tossed and one fair, six-sided die being rolled.

Solution:

$$S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$$

Example: Find the sample space of the experiment of two fair coins being tossed.

Solution:

$$S = \{\text{HH, HT, TT, TH}\}$$

Probability

Example: Let's say that our experiment is rolling a regular, 6-sided die.

The possible outcomes of such an experiment would be rolling a 1, 2, 3, 4, 5 or 6. These outcomes are all *equally likely*.

The sample space is the set of all possible outcomes, thus

$$S = \{1, 2, 3, 4, 5, 6\}$$

A desired event might be rolling a 3.

The probability of rolling a 3 would be expressed as

$$P(3) = \frac{\text{Number of ways to roll a 3}}{\text{Total number of ways a die can be rolled}}$$

$$P(3) = \frac{1}{6}$$

In general, probability can be thought of as the fraction

$$\frac{\text{Possible ways the event can occur}}{\text{Total number of possible outcomes}}$$

The Probability of an Event

If an event E has $n(E)$ equally likely outcomes an its sample space S has $n(S)$ equally likely outcomes, the probability of event E is

$$P(E) = \frac{n(E)}{n(S)}$$

Because the event is a subset of the sample space, the number of outcomes in the event will always be less than or equal to the number in the sample space. Therefore,

$$0 \leq P(E) \leq 1$$

- If $P(E) = 0$, event E cannot occur, and E is called an impossible event.
- If $P(E) = 1$, event E must occur, and E is called a certain event.

Example: A single card is drawn from a standard deck of playing cards.

a) What is the probability of drawing a king?

$$P(K) = \frac{\text{\# ways to draw a king}}{\text{number of possible draws}} = \frac{4}{52} = \frac{1}{13}$$

b) What is the probability of drawing a club?

$$P(\text{club}) = \frac{\text{\# ways to draw a club}}{\text{number of possible draws}} = \frac{13}{52} = \frac{1}{4}$$

Example: Two six-sided dice are tossed.

a) What is the probability that the total is 4?

$$P(4) = \frac{\text{\# rolls that will give a total of 4}}{\text{\# of possible rolls}} = \frac{3}{6 \cdot 6} = \frac{3}{36} = \frac{1}{12}$$

- Use the counting principle to find the number of ways to roll 2 dice.

b) What is the probability that the total is less than 5?

Solution:

1st die	2^{nd} die
1	1
1	2
1	3
2	1
2	2
3	1

$$P(E) = \frac{\text{\# rolls that will give a total} < 5}{\text{\# of possible rolls}} = \frac{6}{36} = \frac{1}{6}$$

Example: A committee of three is to be picked at random from a group of four boys and five girls. What is the probability that the committee will consist entirely of boys?

Solution: $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{{}_4C_3}{{}_9C_3} = \dfrac{4}{84} = \dfrac{1}{21}$

Example: A box contains 3 red marbles, 5 black marbles, and 2 yellow marbles. If a marble is selected at random from the box, what is the probability that it is yellow?

Solution: $P(yellow) = \frac{n(E)}{n(S)} = \frac{2}{10} = \frac{1}{5}$

Note: Probability can be expressed as a fraction, decimal, or percent. For this problem, the probability of drawing a yellow is

$\frac{1}{5}$, 0.2, or 20%

Example: A person draws two cards from a deck of 52 cards. What is the probability that the two cards drawn will both be face cards?

Solution: $P(facecard) = \frac{n(E)}{n(S)} = \frac{{}_{12}C_2}{{}_{52}C_2} = \frac{66}{1326} = \frac{11}{221}$

Example: In a state lottery, a player chooses 6 different numbers from 1 to 41. If these numbers match the 6 numbers drawn by the lottery commission, in any order, the player wins the top prize. What is the probability of winning?

<u>*Solution*</u>: $P(win) = \dfrac{n(E)}{n(S)} = \dfrac{1}{{}_{41}C_6} = \dfrac{1}{4{,}496{,}388}$

<u>Mutually Exclusive Events</u>

<u>**Definition**</u>**:** Two events *A* and *B* (from the same sample space) are <u>mutually exclusive</u> if *A* and *B* have no outcomes in common.

To find the probability that one or the other of 2 mutually exclusive events will occur, we <u>*add*</u> their individual probabilities.

Probability of the Union of Two Events

If *A* and *B* are events in the same sample space, the probability of *A or B* occurring is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

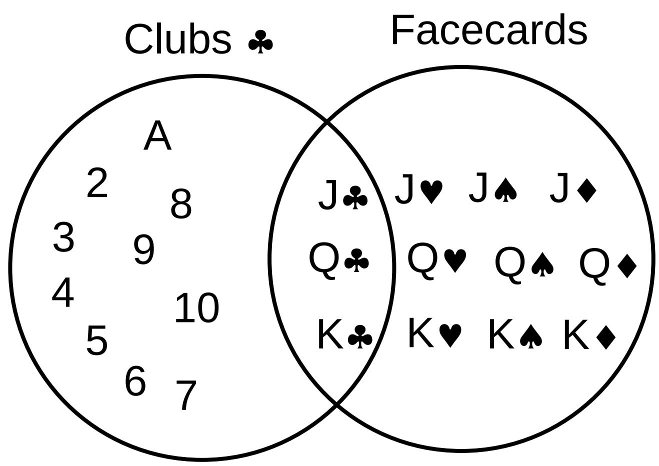
If *A* and *B* are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B)$$

Example: A single card is drawn from a standard deck of playing cards. What is the probability of drawing a club or a face card?

<u>*Solution*</u>: These are not mutually exclusive events.

Look at the Venn diagram:



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} + \frac{12}{52} - \frac{3}{52}$$

$$= \frac{22}{52}$$

$$= \frac{11}{26}$$

If you think about it in more general terms of *P(E),* there 22 draws that will give us our desired outcome (a club or face card). There are 52 possible draws. Therefore,

$$P(E) = \frac{n(E)}{n(S)} = \frac{22}{52} = \frac{11}{26}$$

Example: A box contains 3 red marbles, 5 black marbles, and 2 yellow marbles. If a marble is selected at random from the box, what is the probability that it is either red or black?

Solution:

These are mutually exclusive events, because when you draw a ball, it will be red, black or yellow. It can not be both red and black, or yellow and red, etc.

$$\begin{aligned} P(red \cup black) &= P(red) + P(black) \\ &= \frac{3}{10} + \frac{5}{10} \\ &= \frac{8}{10} \\ &= \frac{4}{5} \end{aligned}$$

If you think about it in more general terms of *P(E),* there 8 draws that will give us our desired outcome (a red or black). There are 10 possible draws. Therefore,

$$P(E) = \frac{n(E)}{n(S)} = \frac{8}{10} = \frac{4}{5}$$

Example: If a regular die is rolled, what is the probability of rolling a 3 or a 5?

$$P(3 \cup 5) = P(3) + P(5) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

Example: If a regular die is rolled, what is the probability of rolling a 3 or an odd?

$$\begin{aligned} P(3 \cup odd) &= P(3) + P(odd) - P(3 \cap odd) \\ &= \frac{1}{6} + \frac{3}{6} - \frac{1}{6} \\ &= \frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$

Example: Two fair, six-sided dice are rolled. What is the probability of getting a total of more than 10?

Solution: There are only 2 totals that are more than 10. They are 11 and 12. So you can think of this as the probability of getting a total of 11 or 12.

$$\begin{aligned} P(11 \cup 12) &= P(11) + P(12) \\ &= \frac{2}{36} + \frac{1}{36} \\ &= \frac{3}{36} = \frac{1}{12} \end{aligned}$$

Independent Events

Definition: Two events are independent if the occurrence of one has no effect on the occurrence of the other.

- An example would be rolling a die and flipping a coin.
- Another example would be rolling the same die twice.

Probablity of Independent Events

If A and B are independent events, the probability that both A and B will occur is

$$P(A \, and \ B) = P(A) \cdot P(B)$$

To find the probability of independent events, we multiply the probabilities of each.

Example: A box contains 3 red marbles, 5 black marbles, and 2 yellow marbles. If two marbles are randomly selected with replacement, what is the probability that both marbles are yellow?

$$\begin{aligned} P(Y \ and \ Y) &= P(Y) \cdot P(Y) \\ &= \frac{2}{10} \cdot \frac{2}{10} \\ &= \frac{4}{100} \\ &= \frac{1}{25} \end{aligned}$$

Example: A box contains 3 red marbles, 5 black marbles, and 2 yellow marbles. If two marbles are randomly selected *without* replacement, what is the probability that both marbles are yellow?

$$\begin{aligned} P(Y_1 \text{ and } Y_2) &= P(Y_1)\cdot P(Y_2) \\ &= \frac{2}{10}\cdot\frac{1}{9} \\ &= \frac{1}{45} \end{aligned}$$

Example: A box contains 3 red marbles, 5 black marbles, and 2 yellow marbles. If two marbles are randomly selected *without* replacement, what is the probability that both marbles are red?

$$\begin{aligned} P(R_1 \text{ and } R_2) &= P(R_1)\cdot P(R_2) \\ &= \frac{3}{10}\cdot\frac{2}{9} \\ &= \frac{1}{15} \end{aligned}$$

Example: A fair coin is tossed three times. What is the probability of getting all heads?

$$P(3Heads) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

Example: In 2000, approximately 65% of the population of the US was 25 years old or older. In a survey, 4 people were chosen at random from the population. What is the probability that all 4 were 25 years old or older?

$$P(25+) = 65\% = 0.65$$

So, the probability of all 4 being 25 or older is

$$P(25+) \cdot P(25+) \cdot P(25+) \cdot P(25+) = (0.65)^4 \approx .1785$$

The Complement of an Event

Suppose you have a group of 30 animals. If *A* represents the set of 10 dogs, then the <u>complement of *A*</u> would be all of the 20 animals who are *not* dogs. The notation would be

A = {dogs}   and   A' = {not dogs}

Look at our sample space of 30 animals in which

A = {dogs} and A' = {not dogs}

Then

$$P(A) = \frac{10}{30} = \frac{1}{3} \quad \text{and} \quad P(A') = \frac{20}{30} = \frac{2}{3}$$

Note that $P(A) + P(A') = 1.$

This makes sense because the 2 events are mutually exclusive and together they account for all of the animals in the sample space.

From this we can use algebra to get

$$P(A') = 1 - P(A)$$

Definition: The <u>complement of an event *A*</u> is the collection of all outcomes in the sample space that are not in *A*.

Probability of a Complement

Let *A* be an event and let *A*' be its complement. If the probability of *A* is *P*(*A*), the probability of the complement is

$$P(A') = 1 - P(A)\ .$$

Example: If the probability of getting a 3 on a spinner is 1/7, what is the probability of *not* getting a 3?

$$P(A') = 1 - P(A)$$
$$P(no\,3) = 1 - P(3)$$
$$P(no\,3) = 1 - \frac{1}{7}$$
$$P(no\,3) = \frac{6}{7}$$

Note: It doesn't matter what the spinner looks like, since we are given the actual probability.

Example: If the probability that it will rain tomorrow has been determined to be 30%, what is the probability that it will *not* rain tomorrow?

solution: 70%

Example: Two fair, six-sided dice are rolled. What is the probability of getting a total of less than or equal to 10?

If A represents the set of sums less than or equal to 10, then A' is the set of the sums greater than 10. Since this is a much smaller set to work with, we will find P(A') and then subtract that number from 1.

$$P(A') = P(11\,or\,12) = \frac{3}{36} = \frac{1}{12}$$

Then

$$P(A) = 1 - P(A') = 1 - \frac{1}{12} = \frac{11}{12}$$